

On the Diophantine Equation $2^a 3^b + 2^c 3^d = 2^e 3^f + 2^g 3^h$

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Abstract

This paper is a continuation of [1], in which I studied Harvey Friedman's problem of whether the function $f(x, y) = x^2 + y^3$ satisfies any identities; however, no knowledge of [1] is necessary to understand this paper. We will break the exponential Diophantine equation $2^a 3^b + 2^c 3^d = 2^e 3^f + 2^g 3^h$ into subcases that are easier to analyze. Then we will solve an equation obtained by imposing a restriction on one of these subcases, after which we will solve a generalization of this equation.

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We will follow the convention that $0 \notin \mathbb{N}$.

A restricted version of Friedman's problem (mentioned in the Abstract) I studied in my paper [1] is related to the solution set of the equation

$$2^a 3^b + 2^c 3^d = 2^e 3^f + 2^g 3^h. \quad (1)$$

The results we will get on this equation, which are still very partial, will not be applied in this paper to Friedman's problem.

Suppose $a_0, b_0, c_0, d_0, e_0, f_0, g_0, h_0$ are nonnegative integers such that

$$2^{a_0} 3^{b_0} + 2^{c_0} 3^{d_0} = 2^{e_0} 3^{f_0} + 2^{g_0} 3^{h_0}. \quad (2)$$

Without loss of generality, we may assume that $\min\{a_0, c_0, e_0, g_0\} = 0 = \min\{b_0, d_0, f_0, h_0\}$, or equivalently $0 \in \{a_0, c_0, e_0, g_0\}$ and $0 \in \{b_0, d_0, f_0, h_0\}$; we can always divide (2) by $2^{\min\{a_0, c_0, e_0, g_0\}} 3^{\min\{b_0, d_0, f_0, h_0\}}$. Suppose that there is exactly one zero in $\{a_0, c_0, e_0, g_0\}$. Then (2) reduces to an equation in which exactly three of its four terms contain a factor of 2, so one side of this resulting equation is divisible by 2 while the other side is not, which is a contradiction. Thus, there must be at least two zeros in $\{a_0, c_0, e_0, g_0\}$ and, by the same reasoning, with the factor 2 replaced by the factor 3, there must be at least two zeros in $\{b_0, d_0, f_0, h_0\}$. Then, depending on which terms of (2) the zeros occur in, we can reduce Equation (1) to 36 cases. However, merging

the cases that are identical up to permutations of the summands, we get the following seven equations:

$$1 + 1 = 2^e 3^f + 2^g 3^h \quad (3)$$

$$1 + 3^d = 2^e + 2^g 3^h \quad (4)$$

$$3^b + 3^d = 2^e + 2^g \quad (5)$$

$$1 + 2^c = 3^f + 2^g 3^h \quad (6)$$

$$1 + 2^c 3^d = 1 + 2^g 3^h \quad (7)$$

$$3^b + 2^c = 1 + 2^g 3^h \quad (8)$$

$$3^b + 2^c = 3^f + 2^g \quad (9)$$

Note, for instance, that in the case $a_0 = b_0 = c_0 = f_0 = 0$, Equation (4) must have at least one solution. The solution to (3) is $(e, f, g, h) = (0, 0, 0, 0)$. The solutions to (7) are $(c, d, g, h) = (s, t, s, t)$ for all nonnegative integers s and t . We now solve (8) subject to the restriction $b = h$, i.e. the equation $2^c - 1 = 3^b(2^g - 1)$. We first prove a few lemmas.

Lemma 1. *Let $p, m, n \in \mathbb{N} \cup \{0\}$ where $p > 1$ and $m > 0$. If $p^m - 1 \mid p^n - 1$, then $m \mid n$.*

Proof. We have $n = qm + r$ where $q, r \in \mathbb{N} \cup \{0\}$ and $0 \leq r < m$. We will prove this lemma by induction on q . For $q = 0$, we have $n < m$, so $p^n - 1 < p^m - 1$, from which it follows that $p^m - 1 \mid p^n - 1 \implies p^n - 1 = 0 \implies n = 0 \implies m \mid n$. Suppose the lemma is true for some q , we will prove it for $q + 1$. Suppose $p^m - 1 \mid p^n - 1 = p^{(q+1)m+r} - 1$. Then $p^m - 1$ divides $p^{(q+1)m+r} - 1 - (p^m - 1) = p^{(q+1)m+r} - p^m = p^m(p^{qm+r} - 1)$. Since $p^m - 1$ and p^m are relatively prime, we have $p^m - 1 \mid p^{qm+r} - 1$. It follows from our inductive hypothesis that $m \mid qm + r$, so $r = 0$ and $n = (q + 1)m$. This completes the induction. \square

Notation 2. Let $n, m, k \in \mathbb{N}$. By $n^k \parallel m$ we will always mean that $n^k \mid m$ and $n^{k+1} \nmid m$.

Lemma 3. *If $k, n \in \mathbb{N}$ where k is odd, then $2^{n+2} \parallel 3^{2^nk} - 1$.*

Proof. We will first prove this claim for $n = 1$. We have $3^{2^k} - 1 = 9^k - 1 = (8+1)^k - 1 = -1 + \sum_{j=0}^k \binom{k}{j} 8^j = \sum_{j=1}^k \binom{k}{j} 8^j = 8 \sum_{j=1}^k \binom{k}{j} 8^{j-1}$, and we see that $\sum_{j=1}^k \binom{k}{j} 8^{j-1}$ is odd because the $j = 1$ term of this sum is odd and all other terms of this sum are even. Now suppose the claim is true for some $n \geq 1$. Then there exists an $l \in \mathbb{N}$ such that l is odd and $3^{2^{n+1}k} - 1 = (3^{2^nk})^2 - 1 = (3^{2^nk} - 1 + 1)^2 - 1 = (2^{n+2}l + 1)^2 - 1 = 2^{2(n+2)}l^2 + 2(2^{n+2}l) + 1 - 1 = 2^{2(n+2)}l^2 + 2(2^{n+2}l) = 2(2^{n+2}l)(2^{n+1}l + 1)$, and we see that $2^{n+1}l + 1$ is odd and $2^{n+3} \parallel 3^{2^{n+1}k} - 1$. This completes the induction. \square

Lemma 4. *If m is odd, then $2^2 \parallel 3^m + 1$.*

Proof. Notice that for $m = 1$ we have $3^m + 1 = 3 + 1 = 4$. Now suppose that the claim is true for some $m \geq 1$, where m is odd. We have $3^{m+2} + 1 = 9 \cdot 3^m + 1 = 9(3^m + 1 - 1) + 1 = 9(4k - 1) + 1 = 36k - 9 + 1 = 36k - 8 = 4(9k - 2)$ where k is odd, and we see that $9k - 2$ is also odd. This completes the induction. \square

Lemma 5. *If $m_1, l \in \mathbb{N}$ and $m_2 \in \mathbb{N} \cup \{0\}$ where $2, 3 \nmid l$, then $3^{m_2+1} \parallel 2^{2^{m_1}3^{m_2}l} - 1$.*

Proof. We will first prove this claim for $m_2 = 0$. Notice that $2^{2^{m_1}l} - 1 = 4^{2^{m_1-1}l} - 1 = (3+1)^{2^{m_1-1}l} - 1 = \sum_{i=1}^{2^{m_1-1}l} \binom{2^{m_1-1}l}{i} 3^i + 1 = \sum_{i=1}^{2^{m_1-1}l} \binom{2^{m_1-1}l}{i} 3^i = \sum_{i=2}^{2^{m_1-1}l} \binom{2^{m_1-1}l}{i} 3^i + 2^{m_1-1}l \cdot 3 = 3(\frac{1}{3} \sum_{i=2}^{2^{m_1-1}l} \binom{2^{m_1-1}l}{i} 3^i + 2^{m_1-1}l)$, and we see that $3 \nmid \frac{1}{3} \sum_{i=2}^{2^{m_1-1}l} \binom{2^{m_1-1}l}{i} 3^i + 2^{m_1-1}l$. Now suppose for some $m_2 \geq 0$ we have $2^{2^{m_1}3^{m_2}l} - 1 = 3^{m_2+1}l_1$ where $3 \nmid l_1$. Then $2^{2^{m_1}3^{m_2}l} = 3^{m_2+1}l_1 + 1 \implies 2^{2^{m_1}3^{m_2+1}l} = (3^{m_2+1}l_1 + 1)^3 = 3^{3(m_2+1)}l_1^3 + 3 \cdot 3^{2(m_2+1)}l_1^2 + 3 \cdot 3^{m_2+1}l_1 + 1$, so $2^{2^{m_1}3^{m_2+1}l} - 1 = 3 \cdot 3^{m_2+1}l_1(3^{2(m_2+1)-1}l_1^2 + 3^{m_2+1}l_1 + 1)$. Since $3 \nmid l_1(3^{2(m_2+1)-1}l_1^2 + 3^{m_2+1}l_1 + 1)$, we have $3^{m_2+2} \parallel 2^{2^{m_1}3^{m_2+1}l} - 1$. This completes the induction. \square

Lemma 6. *If $m_1, l \in \mathbb{N} \cup \{0\}$ where $2, 3 \nmid l$, then $3^{m_1+1} \parallel 2^{3^{m_1}l} + 1$.*

Proof. We will first prove this claim for $m_1 = 0$. Notice that $2^l + 1 = (3 - 1)^l + 1 = \sum_{k=1}^l \binom{l}{k} 3^k (-1)^{l-k} + (-1)^l + 1 = \sum_{k=1}^l \binom{l}{k} 3^k (-1)^{l-k} - 1 + 1 = \sum_{k=1}^l \binom{l}{k} 3^k (-1)^{l-k} = 3 \sum_{k=1}^l \binom{l}{k} 3^{k-1} (-1)^{l-k} = 3(\sum_{k=2}^l \binom{l}{k} 3^{k-1} (-1)^{l-k} + l)$, and we see that $3 \nmid \sum_{k=2}^l \binom{l}{k} 3^{k-1} (-1)^{l-k} + l$. Suppose for some $m_1 \geq 0$ we have $2^{3^{m_1}l} + 1 = 3^{m_1+1}l_1$ where $3 \nmid l_1$. Then we have $2^{3^{m_1+1}l} = 3^{m_1+1}l_1 - 1 \implies 2^{3^{m_1+1}l} = (3^{m_1+1}l_1 - 1)^3 = 3^{3(m_1+1)}l_1^3 - 3 \cdot 3^{2(m_1+1)}l_1^2 + 3 \cdot 3^{m_1+1}l_1 - 1$, so $2^{3^{m_1+1}l} + 1 = 3 \cdot 3^{m_1+1}l_1(3^{2(m_1+1)-1}l_1^2 - 3^{m_1+1}l_1 + 1)$. Since $3 \nmid l_1(3^{2(m_1+1)-1}l_1^2 - 3^{m_1+1}l_1 + 1)$, we have $3^{m_1+2} \parallel 2^{3^{m_1+1}l} + 1$. This completes the induction. \square

Proposition 7. *The only solutions (k, m, n) in the positive integers of the exponential Diophantine equation $3^k(2^m - 1) = 2^n - 1$ are $(1, 1, 2)$ and $(2, 3, 6)$.*

Proof. We know from Lemma 1 that $n = lm$ for some $l \in \mathbb{N}$. Since $k > 0$, we must have $l \geq 2$. Notice that $2^n - 1 = 2^{lm} - 1 = (2^m - 1)(2^{(l-1)m} + 2^{(l-2)m} + \dots + 2^m + 1) = 3^k(2^m - 1)$, so

$$3^k = 2^{(l-1)m} + 2^{(l-2)m} + \dots + 2^m + 1 > 2^m - 1. \quad (10)$$

It follows that $3^{2k} > 3^k(2^m - 1)$, so

$$3^{2k} > 2^n - 1. \quad (11)$$

Now, by Lemma 5 we have $3^k \mid 2^n - 1 \implies n = 2^{m_1}3^{m_2}l_1$ where $m_1, l_1 \in \mathbb{N}$ and $m_2 \in \mathbb{N} \cup \{0\}$ such that $2, 3 \nmid l_1$ and $m_2 \geq k - 1$. Note that $m_2 = k - 1$ if $3^k \parallel 2^n - 1$.

We shall now prove by induction that $3^{2k} < 2^{2^{m_1}3^{k-1}l_1} - 1$ for every $k \geq 3$ and all choices of m_1 and l_1 . It is easy to check that, for all choices of m_1 and l_1 , we have $3^{2k} < 2^{2^{m_1}3^{k-1}l_1} - 1$ for $k = 3$. Suppose we know, for some value of $k \geq 3$, that $3^{2k} < 2^{2^{m_1}3^{k-1}l_1} - 1$, or equivalently $3^{2k} + 1 < 2^{2^{m_1}3^{k-1}l_1}$, for all choices of m_1 and l_1 . Then we have $2^{2^{m_1}3^k l_1} = (2^{2^{m_1}3^{k-1}l_1})^3 > (3^{2k} + 1)^3 > 3^{2(k+1)} + 1$ for all choices of m_1 and l_1 . This completes the induction. If $k \geq 3$ and $3^k \mid 2^n - 1$, then $2^n - 1 = 2^{2^{m_1}3^{m_2}l_1} - 1 \geq 2^{2^{m_1}3^{k-1}l_1} - 1 > 3^{2k}$, contradicting (11). Thus, we see that there are no solutions for $k \geq 3$.

It remains to determine the possible solutions when $k = 1, 2$. Suppose $k = 1$. Then $3^{2k} = 3^2 > 2^n - 1$ implies that $n = 1, 2$, or 3 , so we must have $n = 2$ because n is even,

hence $m = 1$. Suppose $k = 2$. Then by (10) we have $3^2 = 9 > 2^m - 1$, so $m = 1, 2$, or 3 . It is easy to check that the cases $m = 1$ and $m = 2$ do not yield solutions. For $m = 3$, we have $n = 6$. \square

Remark 8. Central to our proof is the fact that, for k sufficiently large, $2^n - 1 = 3^k \cdot q$ implies q must be much larger than 3^k .

We will now solve a more general exponential Diophantine equation using the same ideas in the proof of the preceding proposition. First, we generalize Lemma 5.

Lemma 9. *Let $m \geq 3$ be an odd positive integer. Then the following statements hold.*

1. *If $n \in \mathbb{N}$ is odd, then $m \nmid (m-1)^n - 1$.*
2. *If $m_1, l \in \mathbb{N}$, $m_2 \in \mathbb{N} \cup \{0\}$, and $2, m \nmid l$, then $m^{m_2+1} \parallel (m-1)^{2^{m_1}m^{m_2}l} - 1$.*

Proof. We consider the two cases separately.

1. We have $(m-1)^n - 1 \equiv (-1)^n - 1 \equiv -1 - 1 \equiv -2 \pmod{m}$, from which the conclusion follows.
2. Our proof is by induction on m_2 .

For $m_2 = 0$, we have $(m-1)^{2^{m_1}l} - 1 = \sum_{i=0}^{2^{m_1}l} \binom{2^{m_1}l}{i} m^i (-1)^{2^{m_1}l-i} - 1 = -1 + 1 + \sum_{i=1}^{2^{m_1}l} \binom{2^{m_1}l}{i} m^i (-1)^{2^{m_1}l-i} = \sum_{i=1}^{2^{m_1}l} \binom{2^{m_1}l}{i} m^i (-1)^{2^{m_1}l-i} =$

$$m \left(\left(\sum_{i=2}^{2^{m_1}l} \binom{2^{m_1}l}{i} m^{i-1} (-1)^{2^{m_1}l-i} \right) - 2^{m_1}l \right). \quad (12)$$

Notice that $m \mid \sum_{i=2}^{2^{m_1}l} \binom{2^{m_1}l}{i} m^{i-1} (-1)^{2^{m_1}l-i}$, but $m \nmid 2^{m_1}l$ because $m \nmid l$ and $\gcd(m, 2^{m_1}) = 1$. Thus, $m \nmid \sum_{i=2}^{2^{m_1}l} \binom{2^{m_1}l}{i} m^{i-1} (-1)^{2^{m_1}l-i} - 2^{m_1}l$ and so $m \parallel (m-1)^{2^{m_1}l} - 1$ as desired.

Suppose for some $m_2 \geq 0$ we have $(m-1)^{2^{m_1}m^{m_2}l} - 1 = m^{m_2+1}l_1$ where $m \nmid l_1$. Then $(m-1)^{2^{m_1}m^{m_2+1}l} = m^{m_2+1}l_1 + 1 \implies (m-1)^{2^{m_1}m^{m_2+1}l} = (m^{m_2+1}l_1 + 1)^m = \sum_{i=0}^m \binom{m}{i} (m^{m_2+1}l_1)^i = 1 + \sum_{i=1}^m \binom{m}{i} (m^{m_2+1}l_1)^i$, so $(m-1)^{2^{m_1}m^{m_2+1}l} - 1 = \sum_{i=1}^m \binom{m}{i} (m^{m_2+1}l_1)^i = m \cdot m^{m_2+1}l_1 + \sum_{i=2}^m \binom{m}{i} (m^{m_2+1}l_1)^i = m^{m_2+2}l_1 + \sum_{i=2}^m \binom{m}{i} m^{(m_2+1)i} l_1^i = m^{m_2+2}(l_1 + \sum_{i=2}^m \binom{m}{i} m^{(m_2+1)(i-1)-1} l_1^i)$. By assumption $m \nmid l_1$. However, $m \mid \sum_{i=2}^m \binom{m}{i} m^{(m_2+1)(i-1)-1} l_1^i$ because m clearly divides $\binom{m}{i} m^{(m_2+1)(i-1)-1} l_1^i$ for all $i \geq 3$ and $\binom{m}{2} m^{m_2} l_1^2 = \frac{m(m-1)}{2} m^{m_2} l_1^2 = \frac{m-1}{2} m^{m_2+1} l_1^2$ is divisible by m since $m-1$ is even. It follows that $m \nmid l_1 + \sum_{i=2}^m \binom{m}{i} m^{(m_2+1)(i-1)-1} l_1^i$. This completes the induction. \square

Remark 10. Part 1 of the lemma holds for all integers $m \geq 3$. If m is even, then we can write $m = 2^k p$ where $k, p \in \mathbb{N}$ and $2 \nmid p$, and we can observe (taking $m_1 = k$ and $l = p$) from (12) that $m^2 \mid (m-1)^{2^k p} - 1$. Thus, Part 2 of the lemma is false for even m .

Proposition 11. *The only solutions (k, p, q, n) in the positive integers of the exponential Diophantine equation $(2n+1)^k((2n)^p-1) = (2n)^q-1$ are $(2, 3, 6, 1)$ and $(1, 1, 2, n)$ for all positive n .*

Proof. We know from Lemma 1 that $q = lp$ for some $l \in \mathbb{N}$. Since $k > 0$, we must have $l \geq 2$. Notice that $(2n)^q-1 = (2n)^{lp}-1 = ((2n)^p-1)((2n)^{(l-1)p}+(2n)^{(l-2)p}+\dots+(2n)^p+1) = (2n+1)^k((2n)^p-1)$, so we have $(2n+1)^k = (2n)^{(l-1)p}+(2n)^{(l-2)p}+\dots+(2n)^p+1 > (2n)^p-1$. It follows from

$$(2n+1)^k > (2n)^p-1 \quad (13)$$

that $(2n+1)^{2k} > (2n+1)^k((2n)^p-1)$, so

$$(2n+1)^{2k} > (2n)^q-1. \quad (14)$$

Since $(2n+1)^k \mid (2n)^q-1$, by Lemma 9 we have $q = 2^{m_1}(2n+1)^{m_2}l_1$ where $m_1, l_1 \in \mathbb{N}$ such that $2, 2n+1 \nmid l_1$ and $m_2 \in \mathbb{N} \cup \{0\}$ such that $m_2 \geq k-1$.

We will prove by induction that for every $k \geq 3$ we have $(2n+1)^{2k} < (2n)^{2^{m_1}(2n+1)^{k-1}l_1}-1$ for all choices of n, m_1 , and l_1 . For $k = 3$, observe that $(2n+1)^{2k} < (2n)^{2^{m_1}(2n+1)^{k-1}l_1}-1$ for all choices of n, m_1 , and l_1 ; take $x := 2n+1$, $m_1 = 1 = l_1$ and note that $x^6 < x^{x^2}-1 < ((x-1)^2)^{x^2}-1 = (x-1)^{2x^2}-1$ for all $x \geq 3$. Suppose we know, for some value of $k \geq 3$, that $(2n+1)^{2k} < (2n)^{2^{m_1}(2n+1)^{k-1}l_1}-1 \iff (2n+1)^{2k}+1 < (2n)^{2^{m_1}(2n+1)^{k-1}l_1}$ for all choices of n, m_1 , and l_1 . Then $(2n)^{2^{m_1}(2n+1)^kl_1} > ((2n+1)^{2k}+1)^{2n+1} \geq (2n+1)^{2k(2n+1)}+1 \geq (2n+1)^{2(k+1)}+1$ for all choices of n, m_1 , and l_1 . This completes the induction. If $k \geq 3$ and $(2n+1)^k \mid (2n)^q-1$, then we have $(2n)^q-1 = (2n)^{2^{m_1}(2n+1)^{m_2}l_1}-1 \geq (2n)^{2^{m_1}(2n+1)^{k-1}l_1}-1 > (2n+1)^{2k}$, contradicting (14). Thus, there are no solutions for $k \geq 3$.

It remains to determine the possible solutions when $k = 1, 2$. Consider $k = 1$. Then by (13) we have

$$2n+1 > (2n)^p-1, \quad (15)$$

which for $n = 1$ becomes $4 > 2^p$, which is false for every $p \geq 2$. Notice that as n increases, $(2n)^p-1$ where $p \geq 2$ increases faster than $2n+1$. It follows that (15) is false for every $p \geq 2$, so we must have $p = 1$. For $p = 1$, we have $(2n+1)^k((2n)^p-1) = (2n+1)(2n-1) = (2n)^2-1 = (2n)^q-1 \implies q = 2$, and n can be any positive integer. Consider $k = 2$. Then by (13) we have

$$(2n+1)^2 > (2n)^p-1, \quad (16)$$

which for $n = 1$ becomes $9 > 2^p-1$, which is false for every $p \geq 4$. Notice that as n increases, $(2n)^p-1$ where $p \geq 4$ increases faster than $(2n+1)^2$. It follows that (16) is false for every $p \geq 4$, so we must have $p = 1, 2$, or 3 . If $p = 3$, then by (16) we must have $n = 1$, so $3^2(2^3-1) = 2^q-1 \implies q = 6$. Take $x := 2n$ for simplicity of notation as we check the remaining two cases. Suppose $p = 2$. Then we have $(x+1)^2(x^2-1) = x^q-1 \implies x^4+2x^3-2x-1 = x^q-1 \implies x^4+2x^3-2x = x^q$, so $q \geq 5$. However, it is easy to see that $x^4+2x^3-2x < x^q$ for $q \geq 5$, so there are no solutions for $p = 2$. Suppose $p = 1$. Then $(x+1)^2(x-1) = x^q-1 \implies x^3+x^2-x-1 = x^q-1 \implies x^3+x^2-x = x^q$,

so $q \geq 4$. However, it is easy to see that $x^3 + x^2 - x < x^q$ for $q \geq 4$, so there are no solutions for $p = 1$. \square

Notation 12. For all $n, m \in \mathbb{N}$ where $n \geq m \geq 2$ there exists a unique $k \in \mathbb{N} \cup \{0\}$ such that $m^k \parallel n$, and we will denote this k by $v_m(n)$.

Since Lemma 9 allowed us to solve the Diophantine equation of Proposition 11, a natural question is whether this lemma can be extended to *all* positive integers $m \geq 3$. If this lemma can be extended thus, then we may be able to solve the more general Diophantine equation

$$(m+1)^k(m^p-1) = m^q-1 \quad (17)$$

using arguments similar to those in the proof of Proposition 11. There may be many different such extensions, some of them stronger than others. For the time being, we state one such possible (rather weak) extension for even integers $m > 3$ as the following

Conjecture 13. *There exists a positive integer N such that for all integers $n \geq N$ and all even integers $m > 3$ we have $(m^{v_m((m-1)^n-1)})^2 \leq (m-1)^n - 1$.*

References

- [1] Roger Tian, *Identities of the Function $f(x, y) = x^2 + y^3$* . Preprint, 18 pp., 2009.